MATH 1040: Limits

Finding Limits Graphically

- **NOTATION:** $\lim_{x \to a} f(x) = L$
- Limits tell us what our functions appears to be *approaching* as x closes in around a. (aka: limits don't care about holes!)
- one-sided limits:
  - right-sided limits approach a from the right: $\lim_{x \to a^+} f(x) = L$
  - left-sided limits approach a from the left: $\lim_{x \to a^-} f(x) = L$
  - If $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$, then $\lim_{x \to a} f(x) = L$
  - If $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = M$, then $\lim_{x \to a} f(x)$ DNE

Finding Limits Algebraically

- **Limit Laws:**
  - $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
  - $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$
  - $\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
  - $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ (if $\lim_{x \to a} g(x) \neq 0$)
  - $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x) \right)^n$
  - $\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x) \right)^{1/n}$
  - $\lim_{x \to a} c = c$
- **Direct Substitution:**
  - *always* try direct substitution first when calculating a limit!
  - plug in a into the function. If the result is a number, this is the answer to your limit.
  - $\lim_{x \to 2} (x^2 - 1) = 2^2 - 1 = 4 - 1 = 3$
- **Direct Sub Yields $\frac{0}{0}$**
  - factor
  - multiply by the conjugate
  - find a common denominator
  - expand any powers
  - replace an absolute value with the appropriate part of the piecewise function

Infinite Limits

- $\lim_{x \to a} f(x) = \pm\infty$
- **Direct Sub Yields** $\frac{\#}{0}$
  - Your answer will be $-\infty$ or ∞
  - Simply (factor) the function as much as possible
  - Look at the factor in the denominator that is giving you 0
  - Determine if this factor is approaching a small $+$ or a small $-$ number as x approaches a
  - $\frac{+}{+} = +$, $\frac{-}{-} = +$, $\frac{-}{+} = -$, $\frac{+}{-} = -$
- **Vertical Asymptotes**
  - $x = a$
  - Solve your denominator equal to 0
  - verify that $x = a$ is a vertical asymptote by seeing if $\lim_{x \to a^-} f(x) = \pm\infty$ **OR** $\lim_{x \to a^+} f(x) = \pm\infty$ **OR** $\lim_{x \to a} f(x) = \pm\infty$

Continuity

- For a function to be continuous at $x = a$, we need
  1. $f(a)$ defined
  2. $\lim_{x \to a} f(x)$ exists
  3. $f(a) = \lim_{x \to a} f(x)$
- We have 4 types of discontinuities:
  - removable
    - violates $f(a) = \lim_{x \to a} f(x)$
    - You'll have a removable discontinuity when you cancel out a factor from the numerator and denominator
    - You can fix a removable discontinuity by "filling in the hole" and making $f(a) = \lim_{x \to a} f(x)$
  - jump
    - violates $\lim_{x \to a} f(x)$ exists (b/c $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$)
  - infinite
    - violates $f(a)$ defined and sometimes $\lim_{x \to a} f(x)$ exists
  - oscillating
    - violates $f(a)$ defined and $\lim_{x \to a} f(x)$ exists

Limits at Infinity

- $\lim_{x \to \pm\infty} f(x) = L$
- tell us the *end behavior* of a function
- **Horizontal Asymptotes**
  - If $\lim_{x \to \pm\infty} f(x)$ is a number L, then $y = L$ is a horizontal asymptote.
- Note that we can have infinite limits at infinity: $\lim_{x \to \pm\infty} f(x) = \pm\infty$
- **Polynomials**
  - $\lim_{x \to \pm\infty} x^n = \infty$ when n is even
  - $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$ when n is odd
  - $\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm} a \cdot x^n = \pm\infty$ depending on the degree of x^n (even or odd) and the sign of a
  - $\lim_{x \to \pm\infty} x^{-n} = \lim_{x \to \pm\infty} \frac{1}{x^n} = 0$
- **Rational Functions where numerator and denominator are *polynomials***
  - identify the highest power of x in the denominator
  - divide *every* term in the function by that highest power from the denominator
  - simplify each term
  - take the limit of each term (recall: $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$)

  - If the degree of the top polynomial is less than the degree of the bottom polynomial, $y = 0$ is a horizontal asymptote
  - If the degrees are the same, the horizontal asymptote is the fraction of the leading coefficients
  - If the degree of the top polynomial is greater than the degree of the bottom polynomial, there are no horizontal asymptotes
  - If the degree of the top polynomial is exactly one more than the degree of the bottom polynomial, there are no horizontal asymptotes, but there is a slant asymptote
    - ☐ Find the equation of a slant asymptote by long polynomial division
- **Rational Functions where numerator and denominator may *not* be polynomials**
  - identify the highest power of x in the denominator
    - ☐ Is the power even or odd when pulled outside of the radical?
    - ☐ If even outside of the radical, that power of x will always be positive.
    - ☐ If odd out outside of the radical, that power of x depends on if $x \to -\infty$ or $x \to \infty$.
  - divide *every* term in the function by that highest power from the denominator
    - ☐ Underneath any radicals, make sure you are dividing each term by the power of x that is appropriate for within the radical as it is outside the radical
    - ☐ Outside the radical, divide by the appropriate sign and power on x
  - simplify each term
  - take the limit of each term (recall: $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$)

MATH 1040: Derivatives

The Limit Definition of a Derivative

- **derivative (as a function) = slope of tangent line *anywhere* on the curve**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Derivative Notation**: y', $f'(x)$, f', $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$
- If we graph the derivative function, we are simply graphing the *slopes of the tangent lines* to our orginal function

function graph	**derivative graph**
increasing	positive (above x-axis)
decreasing	negative (below x-axis)
smooth min/max	zero (cross x-axis)
constant	zero (over an interval)
linear	constant
quadratic	linear

- A function will not be differentiable at a point (can't find derivative) if it has the following at that point:
  - discontinuity
  - sharp point (corner or cusp)
  - vertical tangent (supa steeeeep)
- If a function is differentiable at a, then it is continuous at a.

EQUATION OF A TANGENT LINE

- We need three things to find the equation of a line:
  - slope (m)
  - x-point (x_1)
  - y-point (y_1)
- If we are finding the equation of a tangent line, the **slope of the tangent line** will be the **derivative**.
- To find the slope of the tangent line at our x_1 value, we plug in x_1 into the derivative.
  - $m_{tan}\Big|_{x_1} = f'(x_1)$
- To find the y_1 value, we plug in our x_1 value into the original function
  - $y_1 = f(x_1)$
- To find the slope of a **normal** line, first find the slope of the tangent line, then flip and negate it.
  - $m_{norm} = -\frac{1}{m_{tan}}$

Derivative Rules

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(cf(x)) = cf'(x)$
- $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
- $\frac{d}{dx}(e^x) = e^x$

- We can rewrite how functions look so we can use power rule:
  - $\frac{1}{x^n} = x^{-n}$
  - $\sqrt[m]{x^n} = x^{n/m}$
- a function has a *horizontal tangent line* when $f'(x) = 0$ because the slope of a horizontal tangent line is zero
- We can find higher order derivatives by taking the derivative of the previous derivative.
  - second derivative: y''
  - third derivative: y'''
  - fourth derivative: $y^{(4)}$
  - n^{th} derivative: $y^{(n)}$

Product Rule and Quotient Rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \qquad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

- Sometimes it's easier to simplify your function first before trying these rules.

Derivatives of Trig Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

- Derivatives of $\sin x$ and $\cos x$ work in a cycle
  - If you want to find the n^{th} derivative of something in this cycle, divide n by 4, find your remainder, then count your remainder away from your original function

Applications of Derivatives

- Physics
  - position: $s(t)$ ft
  - velocity: $v(t) = s'(t)$ ft/s
  - acceleration: $a(t) = v'(t) = s''(t)$ ft/s^2
  - speed $= |v(t)|$ ft/s
  - To find an object's maximum height:
    1. First find *when* the object reaches its max height by solving where $v(t) = 0$.
    2. Then, find the max height by plugging the time you found in Step 1 into your position function, $s(t)$.
  - To find the velocity in which an object strikes the ground:
    1. First find *when* an object strikes the ground by solving where $s(t) = 0$.
    2. Then, find the velocity it strikes the ground by plugging the time you found in Step 1 into your velocity function, $v(t)$.
  - Velocity's sign signifies direction. An object possibly changes direction when $v(t) = 0$
    * If $v(t) < 0$, the object is moving down or to the left
    * If $v(t) > 0$, the object is moving up or to the right

Chain Rule

- Chain Rule tells us how to take the *derivative of a composition of functions*
  - $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$
- Chain Rule Steps
  1. First, identify your outer and inner functions.
  2. Take the derivative of your first, leaving your inside function ***unchanged***.
     - $f'(g(x))$
  3. Multiply by the derivative of the inside function
     - $f'(g(x)) \cdot g'(x)$
- You can use Chain Rule as many times as needed to ensure you take the derivative of your entire composition of functions.

MATH 1070: Derivatives and Applications

Implicit Differentiation

- Explicit vs Implicit
  - **Explicit Form:** An equation that is solved for a single variable; $y = x^2 + 2x - 1$
  - **Implicit Form:** An equation that is not solved for one variable; $xy + x^2 = \sin(y) + 6x$
- Implicit Differentiation Steps
  1. Take the derivative of **both sides** with respect to x, $\frac{d}{dx}$.
  2. Solve the resulting equation for your derivative notation, y' or $\frac{dy}{dx}$
- Chain Rule Reminders

$$\frac{d}{dx}(y) = y' = \frac{dy}{dx} \qquad \frac{d}{dx}(y^2) = 2yy' = 2y\frac{dy}{dx} \qquad \frac{d}{dx}(y^3) = 3y^2y' = 3y^2\frac{dy}{dx}$$

Derivatives of Logarithmic and Exponential Functions

$\frac{d}{dx}(\log_b(g(x)) = \frac{1}{g(x)\ln(b)} \cdot g'(x)$	$\frac{d}{dx}b^{g(x)} = b^{g(x)} \cdot g'(x) \cdot \ln(b)$
$\frac{d}{dx}(\ln(g(x)) = \frac{1}{g(x)} \cdot g'(x)$	$\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot g'(x)$

Logarithmic Differentiation

- You **MUST** use Logarithmic Differentiation when you have a variable in your **base AND** a variable in your **exponent**.
- You can use Logarithmic Differentiation for taking the derivative of any function, though.
- Laws of Logarithms

$$\ln(xy) = \ln(x) + \ln(y) \qquad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y) \qquad \ln(x^r) = r \cdot \ln(x)$$

- Laws of Logarithms Steps
  1. Take the natural log (ln) of **both sides** of the equation.
  2. Use Law of Logarithms to simplify the result.
  3. Use implicit differentiation to take the derivative of **both sides**.
  4. Isolate your derivative; y' or $\frac{dy}{dx}$.
  5. Substitute y back into the result.

Inverse Trig Functions

- Inverse Trig Functions take in a **number** as their input and give an **angle** as their output.
- Consider the restricted domain of each trig function to verify the existence of its inverse.

Function	Domain	Range
$\sin^{-1}(x)$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$\cos^{-1}(x)$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$(-\frac{\pi}{2},\frac{\pi}{2})$
$\cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$
$\sec^{-1}(x)$	$(-\infty,-1]\cup[1,\infty)$	$[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
$\csc^{-1}(x)$	$(-\infty,-1]\cup[1,\infty)$	$[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}$

- **Example:** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$ because $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$ on the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$.

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\left(\sin^{-1}(x)\right)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1}(x)\right)=\frac{-1}{\sqrt{1-x^2}} \qquad \text{for } -1<x<1$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right)=\frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1}(x)\right)=\frac{-1}{1+x^2} \qquad \text{for } -\infty, x, \infty$$

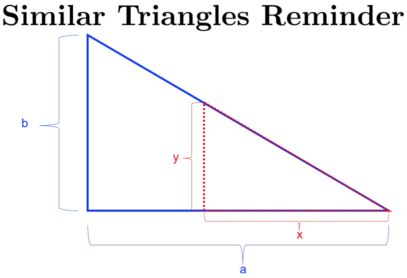
$$\frac{d}{dx}\left(\sec^{-1}(x)\right)=\frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1}(x)\right)=\frac{-1}{|x|\sqrt{x^2-1}} \qquad \text{for } |x|>1$$

- **DO NOT FORGET ABOUT CHAIN RULE!**
  - $\frac{d}{dx}\left(\sin^{-1}(g(x))\right)=\frac{1}{\sqrt{1-(g(x))^2}}\cdot g'(x)=\frac{g'(x)}{\sqrt{1-(g(x))^2}}$

Related Rates

1. Read the problem, make a sketch, write down given information, and identify the rate to find.
2. Write one or more equations that express the basic relationship among the variables.
3. Introduce rates of change by differentiating the appropriate equation with respect to time, t.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable; i.e. does your answer have the correct sign? It helps to state our final answer as a complete sentence.

Similar Triangles Reminder



$$\frac{b}{a}=\frac{y}{x}$$

Maxima and Minima

- **Absolute Extrema**
  - absolute max: highest function value over the entire domain
  - absolute min: lowest function value over the entire domain
- **Extreme Value Theorem:** A function that is continuous on a closed interval $[a, b]$ has an absolute max and an absolute min on $[a, b]$
- **Local Extrema**
  - local max: highest function value in a small neighborhood
  - local min: lowest function value in a small neighborhood
  - Note: *local extrema cannot occur at endpoints*
- **Critical Point:** a point c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist
- **Closed Interval Method:** an algebraic way to locate absolute extrema on a continuous function over $[a, b]$
  1. Find the critical points of f
  2. Evaluate f at the critical points and the endpoints
  3. Your absolute max is the largest function value and your absolute min is the smallest function value

Rolle's Theorem and MVT

- **Rolle's Theorem:** If a function is **continuous** on $[a, b]$, **differentiable** on (a, b), and $\mathbf{f(a) = f(b)}$, then there is a point c in (a, b) where $f'(c) = 0$.
  - Helps us find where we have horizontal tangent lines
- **Mean Value Theorem:** If a function is **continuous** on $[a, b]$ and **differentiable** on (a, b), then there is a point c in (a, b) where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
  - Helps us find where the slope of any tangent lines is the same as the slope of the secant line between a and b

What the First Derivatives Tell Us

- **Intervals of Increase/Decrease**
  - $f'(x) > 0$ tells us that $f(x)$ is *increasing*
  - $f'(x) < 0$ tells us that $f(x)$ is *decreasing*
- **Increasing/Decreasing Test**
  1. Find all x-values where $f'(x) = 0$ or $f'(x)$ does not exist
  2. Make a sign chart for $f'(x)$ with the sign of the first derivative between these x-values
  3. $f'(x)$ + means that $f(x)$ is increasing on that interval; $f'(x)$ - means that $f(x)$ is decreasing on that interval
- **The First Derivative Test for Local Extrema**
  - We have a *local min* at a critical point c if $f'(x)$ changes from $-$ to $+$ at c
  - We have a *local max* at a critical point c if $f'(x)$ changes from $+$ to. $-$ at c
  - There is no local extrema at c if $f'(x)$ does not change sign at c
- **One Local Extrema $\implies$ Absolute Extrema**: If a function has *only one* local extrema, then that local extrema is an absolute extrema

What the Second Derivatives Tell Us

- **Intervals of Concavity**
  - $f''(x) > 0$ tells us that $f(x)$ is *concave up* (CCU)
  - $f''(x) < 0$ tells us that $f(x)$ is *concave down* (CCD)
- **Concavity Test**
  1. Find all x-values where $f''(x) = 0$ or $f''(x)$ does not exist
  2. Make a sign chart for $f''(x)$ with the sign of the first derivative between these x-values
  3. $f''(x)$ + means that $f(x)$ is CCU on that interval; $f''(x)$ - means that $f(x)$ is CCD on that interval
- **Inflection Point:** a point on $f(x)$ where the *concavity changes*
- **Second Derivative Test for Local Extrema**:
  - We have a *local min* at a critical point c if $f''(c) > 0$
  - We have a *local max* at a critical point c if $f''(c) < 0$
  - The test is inconclusive if $f''(c) = 0$

<u>**Sketching Functions**</u>

1. **Domain**
   - Identify the domain of the function on the given interval by looking for values that must be excluded from our knowledge of the parent functions.
   - i.e. the denominator $= 0$ for rational functions

2. **Intercepts**
   - y-intercept: Set $x = 0$ and solve for y (*crosses y-axis*)
   - x-intercept: Set $y = 0$ and solve for x (*crosses x-axis*)

3. **Symmetry**
   - Evaluate $f(-x)$.
     - **EVEN:** symmetric about y-axis; $f(-x) = f(x)$
     - **ODD:** symmetric about origin; $f(-x) = -f(x)$
   - Look for periodic functions too (i.e. $\sin x$, $\cos x$)

4. **Asymptotes**
   - **Horizontal Asymptotes:** Evaluate $\lim_{x\to\pm\infty} f(x)$.
     - If an answer is a number L, there is a horizontal asymptote, $y = L$.
   - **Vertical Asymptotes:** Solve denominator $= 0$.
     - $x = a$ could be a vertical asymptote, we should verify with limits.
     - Verify Vertical Asymptote: Evalute $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.
       If answer is ∞ or $-\infty$, then vertical asymptotes is $x = a$.
   - **Slant Asymptote:** If the degree of the numerator is *one more* than the degree of denominator, we will have a slant asymptote.
     - Use polynomial long division to find the equation of the slant asymptote.

5. **Intervals of Increase and Decrease**
   - find the first derivative, y'
   - Make a sign chart for y'
   - $y' < 0$: y is decreasing
   - $y' > 0$: y is increasing

6. **Local Extrema**
   - local min location: y' changes from $-$ to $+$
   - local max location: y' changes from $+$ to $-$
   - local extrema value: evaluate y at these points

7. **Concavity and Points of Inflection**
   - find the second derivative, y''
   - make a sign chart for y''
   - $y'' < 0$: y concave down (CCD)
   - $y'' > 0$: y concave up (CCU)
   - inflection points: where the sign changes
   - find the y values at these points for graphing purposes

8. **Sketch the Function:** Using the information found in 1-7, sketch the function

Optimization Problems

- **Objective Function:** what you want to "minimize" or "maximize" or make "smallest" or "largest"
- **Constraints:** limitations to follow

1. Read the problem carefully and define all variables.
   - What quantities are given?
   - What objective function are we trying to minimize or maximize?
   - What constraints are given?
2. Draw a diagram if possible.
3. Express the function to be optimized in terms of the variables.
4. Express the constraints in terms of the variables.
5. Use the constraints to eliminate all but one variable of the objective function.
6. Find the domain of the objective function.
7. Use calculus to find the absolute extrema of the objective function on the domain.
   - Find the critical points. ($f'(x) = 0$ or $f'(x)$ DNE)
8. Verify that you have found the location of the absolute extrema. Use can use:
   - Closed Interval Method (Extreme Value Theorem)
     - Find the critical points of f
     - Evaluate f at the critical points and the endpoint
     - Establish the absolute max or absolute min from the function values
   - First Derivative Test
     - Find the critical points, c, of f
     - Draw a sign chart for f'
     - If f' goes from $-$ to $+$ at c, you have an absolute min
     - If f' goes from $+$ to $-$ at c, you have an absolute max.
   - Second Derivative Test
     - Determine the local extreme values of f
     - If $f(c)$ is a local min and $f''(x) > 0$, then we have an absolute min at c
     - If $f(c)$ is a local max and $f''(x) < 0$, then we have an absolute max at c
9. Write a summary sentence answering the question.

Linear Approximations and Differentials

- **Linear Approximation to f at a:** $L(x) = f(a) + f'(a)(x - a)$
  - **Underestimate:** $f''(a) > 0$; linear approximation is below the curve
  - **Overestimate:** $f''(a) < 0$; linear approximation is above the curve
- **Differential:** $dy = f'(x)dx$
  - How much a small change in input dx affects the change in output dy

Limits with Indeterminate Forms

- **Quotient Indeterminate Forms** ($\frac{0}{0}$, $\frac{\infty}{\infty}$)
  - We can use L'Hopital's Rule to avoid these quotient indeterminate forms
  - **L'Hopital's Rule:** $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} \overset{\text{L}}{=} \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$
- **Related Indeterminate Forms** ($\infty\cdot 0$,$\infty-\infty$)
  - Use algebra to rearrange your function to appear as a single quotient. Some options...
    - move one function to the denominator
    - find a common denominator
  - Use L'Hopital's Rule as needed on any quotient indeterminate forms.
- **Indeterminate Powers** (1^∞, 0^0, ∞^0)
  - Rewrite your function using e^x and $\ln(x)$, the fact that they are inverses, and laws of logarithms.
  $$f(x)^{g(x)} = e^{\ln(f(x)^{g(x)})} = e^{g(x)\cdot\ln(f(x))}$$
  - First, take the limit of just the exponent, $g(x)\cdot\ln(f(x))$, of your rewritten function.
  $$\lim_{x\to a}\Big(g(x)\cdot\ln(f(x))\Big) = \#$$
    - You may need to use Related Indeterminate Form methods and L'Hopital's Rule.
    - You will get a numerical answer.
  - Exponentiate the value you found from Step 3 with base e. This is your final answer.
  $$\lim_{x\to a} f(x)^{g(x)} = e^{\#}$$

Antiderivatives and Indefinite Integrals

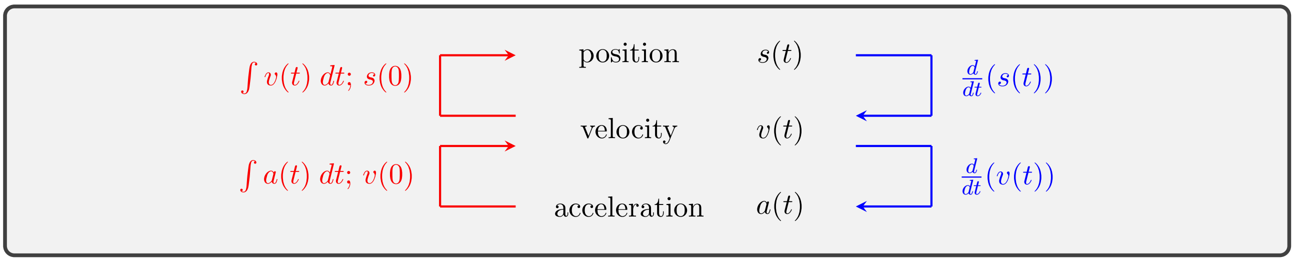
- **Antiderivative:** If $F(x)$ is an antiderivative of $f(x)$ on an interval I, then all the antiderivatives of $f(x)$ on I have the form $F(x)+C$ where C is an arbitrary constant.
- When taking antiderivatives think, **"What can I take the derivative of to get what I see here?"**
- We can check our work with differentiation.

$$\int f(x)\,dx = F(x)+C$$

- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$; for $n\neq -1$
  - Add one to our exponent
  - Divide by the new exponent
- $\displaystyle\int cf(x)\,dx = c\int f(x)$
- $\displaystyle\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x|+C$

Differential Equations/Initial Value Problems

- If we are given a function and an initial value, we can find the value of our constant $+C$
  - Find the antiderivative of the given function.
  - Find the value for $+C$ by plugging in the given initial value into your antiderivative.
- We can work backwards from acceleration and find our velocity and position functions given an initial velocity and initial position.



Sigma (Summation) Notation

- If $a_m, a_{m+1}, \cdots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

- **Summation Properties:** Let n be a positive integer and c be a real number.
  - $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$
  - $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- **Sums of Infinite Powers:** (don't have to memorize) Let n be a positive integer.
  - $\sum_{k=1}^{n} c = cn$
  - $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$
  - $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$
  - $\sum_{k=1}^{n} k^3 = \dfrac{n^2(n+1)^2}{4}$

Riemann Sums

- We can approximate the area under a curve by drawing rectangles between our curve and the x-axis and adding up the areas of all the rectangles.

- **Riemann Sum:** Suppose f is defined on a closed interval $[a, b]$, which is divided into n sub-intevals of equal length Δx. If x_k^* is a point on the k^{th} sub-interval $[x_{k-1}, x_k]$, for $k = 1, 2, \cdots, n$, then the Riemann Sum for f on $[a, b]$ is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

  - **width:** $\Delta x = \dfrac{b-a}{n}$; this is the width of each rectangle were n is the number of rectangles used
  - **height:** $f(x_k^*)$; this is the height of the k^{th} rectangle
  - **area:** $f(x_k^*) \cdot \Delta x$; this is the area of the k^{th} rectangle
  - **Left Riemann Sum:** $x_k^* = a + (k-1)\Delta x$
  - **Right Riemann Sum:** $x_k^* = a + k\Delta x$
  - **Midpoint Riemann Sum:** $x_k^* = a + (k - 1/2)\Delta x$

Definite Integrals

- Definite Integrals measure the *net area*
  - net area = (area above x-axis) - (area below x-axis)

- We can find the definite integral between a and b by taking the limit of a Riemann Sum. This limit as $n \to \infty$ is pushing the number of rectangles under our curve to ∞.

$$\boxed{\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x}$$

- **Properties of Definite Integrals:**

  - $\displaystyle\int_a^a f(x)\, dx = 0$
  - $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
  - $\displaystyle\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$
  - $\displaystyle\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
  - $\displaystyle\int_a^b f(x)\, dx = \int_a^p f(x)\, dx + \int_p^b f(x)\, dx$
  - $\displaystyle\int_a^b |f(x)| dx$ finds total area

The Fundamental Theorem of Calculus (Part I)

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_0^x f(t)\, dt, \qquad \text{for } a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

Given Integral	How to Get Derivative
$f(x) = \int_a^x f(t)dt$	plug in x for t
$f(x) = \int_x^a f(t)dt$	reverse limits of integration, multiply by -1 plug in x for t
$f(x) = \int_a^{g(x)} f(t)dt$	plug in $g(x)$ for t, multiply by $g'(x)$
$f(x) = \int_{g(x)}^a f(t)dt$	reverse limits of integration, multiply by -1 plug in $g(x)$ for t, multiply by $g'(x)$
$f(x) = \int_{g(x)}^{h(x)} f(t)dt$	split the limits of integration as $\int_{g(x)}^0 f(t)dt + \int_0^{h(x)} f(t)dt$ reverse limits of integration of $\int_{g(x)}^0 f(t)dt$ and multiply by -1 plug $g(x)$ and $h(x)$ in for t and multiply by $g'(x)$ and $h'(x)$

The Fundamental Theorem of Calculus (Part II)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(x)\Big|_a^b = F(b) - F(a)$$

Symmetry of Functions

- If f is **even**, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- If f is **odd**, then $\int_{-a}^{a} f(x)\,dx = 0$

Mean Value Theorem for Integrals

- **Average Value**
  - The average value of an integrable function f on the interval $[a, b]$ is $\bar{f} = \dfrac{1}{b-a}\int_{a}^{b} f(x)\,dx$
  - The average value is the height of the rectangle with base $[a, b]$ that has the same net area as the region bounded by the graph of f and the x-axis on the same interval $[a, b]$
- **MVT for Integrals:** Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) where

$$f(c) = \bar{f} = \frac{1}{b-a}\int_{a}^{b} f(t)\,dt$$

u-Substitution: Indefinite Integrals

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

1. Identify an inner function $u = g(x)$ such that a constant multiple of $u' = g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\, dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

u-Substitution: Definite Integrals

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du = F(g(b)) - F(g(a))$$

1. Identify an inner function $u = g(x)$ such that a constant multiple of $u' = g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\, dx$ in the integral as well as $g(a) = a$ and $g(b) = b$ for your limits of integration.
3. Evaluate the new definite integral with respect to u with your new limits of integration.